21-241 Exam 2 Solutions

1. Note that this problem spans two pages. Determine whether the following statements are true or false. If a statement is true, prove it; if it is false, provide a counterexample.

   (a) Every Markov matrix has a unique steady state vector.

   False. The matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a Markov matrix in which every probability vector $\mathbf{v}$ is a steady state vector, since $M\mathbf{v} = I\mathbf{v} = \mathbf{v}$.

   (b) If A is an $n \times n$ diagonalizable matrix having an eigenvalue of $\lambda = 1$ with algebraic multiplicity n, then $A = I$.

   True. Since A is diagonalizable, $A = PDP^{-1}$, where D is the diagonal matrix whose diagonal entries are the eigenvalues of A, which are all 1. Hence $D = I$ and so $A = PDP^{-1} = PIP^{-1} = PP^{-1} = I$.

   (c) If A is an $n \times n$ matrix with orthogonal columns, then $||A\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$.

   False. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then A has orthogonal columns. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

   $$||A\mathbf{x}|| = \left|\left|\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right|\right| = \sqrt{2} \text{ and } ||\mathbf{x}|| = \left|\left|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right|\right| = 1.$$

   Hence $||A\mathbf{x}|| \neq ||\mathbf{x}||$.

   (d) Every orthogonally diagonalizable matrix is symmetric.

   True. Let $A = QDQ^T$ be an orthogonal diagonalization. Then

   $$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$$

   Hence A is symmetric.

2. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

   (a) Use the Gram-Schmidt process to find an orthonormal basis of W.

   First we find an orthogonal basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ and then we normalize the vectors to obtain an orthonormal basis.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Now normalizing the vectors gives $\mathbf{q}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Hence the orthonormal basis of W is $\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

(b) Explain how the four fundamental subspaces can be used to find a basis for the orthogonal complement of W (no computation required).

Let A be the matrix whose columns are the basis vectors of W. Then $C(A) = W$ and so $W^\perp = C(A)^\perp = N(A^T)$. Hence to find a basis of the orthogonal complement of W, we can find a basis of $N(A^T)$.

(c) Determine the projection of $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ onto $W^\perp$.

[Hint: there is a quick way to do this and a long way to do this]

Note that $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is the second orthogonal basis vector we found while using the Gram-Schmidt process in part (a). Hence it is in W. By definition, it is orthogonal to $W^\perp$. Thus its projection onto $W^\perp$ is $\mathbf{0}$.

3. Let A be a 4×4 matrix with QR factorization $A = QR$, where $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

(a) Determine the possible values of the determinant of A.

First note that since R is upper triangular, $\det R = (1)(1)(3)(-2) = -6$. Since Q is orthogonal, $\det Q = \pm 1$. Hence $\det A = \det(QR) = \det Q \det R = \pm 6$.

(b) Compute $A^T A$.

Since Q is orthogonal, $Q^TQ = I$. Hence

$$A^TA = (QR)^T(QR) = R^TQ^TQR = R^TR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ -1 & 2 & 14 & -2 \\ 0 & -1 & -2 & 5 \end{bmatrix}.$$

(c) Is A^TA positive definite?

Yes. Since $\det A \neq 0$, A is invertible and so A has linearly independent columns (equivalently, since A has a QR factorization, it must have linearly independent columns). Hence A^TA is invertible, implying that it has no zero eigenvalues. It now follows the eigenvalues of A^TA are positive and so A^TA is positive definite.

(d) What does the Spectral Theorem tell you about A^TA?

Since A^TA is symmetric, the Spectral Theorem tells us that A^TA is orthogonally diagonalizable; that is, there exists an orthogonal matrix Q and a diagonal matrix D such that $A^TA = QDQ^T$.

4. Note that this problem spans two pages. Let $A = \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & 1 \end{bmatrix}$.

(a) Determine the eigenvalues of A.

Since A is upper triangular, its eigenvalues are its diagonal entries, $\frac{1}{2}$ and 1.

(b) Find a basis for each eigenspace of A.

$\underline{E_{\frac{1}{2}}(A)}$: We must find a basis for $N(A - \frac{1}{2}I)$. Reducing $A - \frac{1}{2}I$ to RREF gives:

$$A - \frac{1}{2}I = \begin{bmatrix} 0 & 3 \\ 0 & 1/2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence x_1 is the free variable and we have the general solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Hence $E_{\frac{1}{2}}(A)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

$\underline{E_1(A)}$: We must find a basis for $N(A - I)$. Reducing $A - I$ to RREF gives:

$$A - I = \begin{bmatrix} -1/2 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -6 \\ 0 & 0 \end{bmatrix}$$

Hence x_2 is the free variable and we have the general solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \end{bmatrix}$.

Hence $E_{\frac{1}{2}}(A)$ has basis $\left\{ \begin{bmatrix} 6 \\ 1 \end{bmatrix} \right\}$.

(c) Diagonalize A.

Using the eigenvalues and bases found above, we have the diagonalization

$$A = PDP^{-1} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}^{-1}.$$

(d) Let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Write a formula for $A^k\mathbf{x}$ as a single vector.

$$\begin{aligned} A^k\mathbf{x} = (PDP^{-1})^k\mathbf{x} = PD^kP^{-1}\mathbf{x} &= \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1/2)^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1/2)^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1/2)^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a - 6b \\ b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1/2)^k(a - 6b) \\ b \end{bmatrix} \\ &= \begin{bmatrix} (1/2)^k(a - 6b) + 6b \\ b \end{bmatrix} \end{aligned}$$

(e) Let $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Show that $A^k\mathbf{x}$ approaches one of the eigenspaces as $k \to \infty$. Which eigenspace is it?

As $k \to \infty$, $A^k\mathbf{x} = \begin{bmatrix} (1/2)^k(a - 6b) + 6b \\ b \end{bmatrix} \to \begin{bmatrix} 6b \\ b \end{bmatrix}$. Since this vector is a multiple of $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$, which forms a basis for $E_1(A)$, we have that $A^k\mathbf{x}$ approaches the eigenspace $E_1(A)$ as $k \to \infty$.